Particle Accelarators, 1970, Vol. 1, pp. 269–270

SMOOTHING AND DIFFERENTIAL ANALYSIS OF MAGNETIC FIELD AND GRADIENT MEASUREMENT

JONATHAN D. YOUNG

Lawrence Radiation Laboratory, University of California, Berkeley, California, USA

We discuss mini-max fifth-degree smoothing and modified quintic spline fitting of a set of median plane magnetic field measurements consisting of

(1) the vertical component as a function of one variable,

(2) the first derivative thereof.

We assume that the measurements have been made for at least five uniformly spaced values of the argument.

In particular the application of this process to measurements made for the field in the Bevatron (Berkeley) are discussed.

1. INTRODUCTION

Previously[1] we described the use of cubic spline fitting for interpolation and approximate differentiation of magnetic field measurements. We considered the possibility that the vertical component of the field was expressible as the product of a function of radius and a function of azimuth:

$$B(\theta, R, 0) = g(\theta) \cdot f(R).$$

Where this separation of variables is possible we may think of $f(R)$ as an 'average' of B over the range for which $g(\theta)$ is defined and think of $g(\theta)$ as expressing the departure from this 'average'.

We now consider that for some range of R we may be able also to measure the first derivative of f. (This was the case in the Bevatron for 15 in. on either side of the center radius of 599.38 in.) Measurements of both f and f' make possible a higher order of approximation than that previously[1] described.

Since the measurements of f and f' were made independently and were subject to different measurement errors, the possibility of inconsistency arose and some type of smoothing of the combined data was indicated. Mini-max smoothing[2] as described in the next section is employed. Modified quintic spline fitting (described later) may then be applied to the 'smoothed' data.

We require measurements of $f(R)$ and $f'(R)$ for at least five distinct uniformly spaced values of R. To distinguish between measured values and smoothed values we shall hereafter denote the former by f^* and f'^* and the latter simply by f and f'.

2. MINI-MAX SMOOTHING

For a set of measurements, $(R_i, f_i^*, f_i'^*)$, where $i = 1, n$ with $n \geqslant 5$, we assume that each f_i^* is subject to a possible measurement error, d_i, and each $f_i'^*$ to a possible error, e_i. Note that $e_i \geqslant 0$ and $d_i \geqslant 0$, but that neither the e_i or the d_i need be uniform in i. We have

$$\left.\begin{aligned} f_i &\geqslant f_i^* - d_i \\ f_i &\leqslant f_i^* + d_i \\ f_i' &\geqslant f_i'^* - e_i \\ f_i' &\leqslant f_i'^* + e_i \end{aligned}\right\} i = 1, n, \tag{A}$$

which establishes for each f_i and f_i' upper and lower bounds.

Any three consecutive values of f^* and f'^* are sufficient to determine a quintic. We should like to obtain smooth values for f and f' so that any *four* consecutive values would be 'as nearly as possible' those of a quintic. If values $f_i, f_{i+1}, f_{i+2}, f_{i+3}, f_i', f_{i+1}', f_{i+2}'$, and f_{i+3}' were correct for a quintic (with uniform steps h, in, R) we should have

$$3f_i - 3f_{i+1} - 3f_{i+2} + 3f_{i+3} + hf_i' + 3hf_{i-1}' - 3hf_{i+2}' - hf_{i+3}' \equiv \epsilon_i = 0$$

for $i = 1, n-3$. Since it is unlikely that all ϵ_i can be made zero within the restraints (A), we let

$$f_\epsilon = \max_i |\epsilon_i| \quad \text{for} \quad i = 1, n-3$$

or

$$|3f_i - 3f_{i+1} - 3f_{i+2} + 3f_{i+3} + h(f_i' + 3f_{i+1}' - 3f_{i+2}' - f_{i+3}')| \leqslant f_\epsilon \quad \text{for} \quad i = 1, n-3. \tag{B}$$

We now seek to minimize f_ϵ subject to the restraints

(A) and (B). The linear model thus posed is the dual of a linear program[2] which can be solved by the Simplex algorithm.[3] The latter provides, in addition to its primary solution (in which we have little interest), the desired values for its dual variables, namely f_i and f_i' for $i = 1, n$. These smoothed values constitute the data to which we apply modified quintic spine fitting as described in the next sector.

3. MODIFIED QUINTIC SPLINE FITTING

Modified quintic spline fitting is described in more detail elsewhere.[4] Briefly, for the data

$$(R_i, f_i, f_i') \quad \text{where} \quad i = 1, n$$

we seek a function $p(R)$ defined on $[R_1, R_n]$ with the following properties:

(1) exact fitting $p_i = f_i$, $p_i' = f_i'$ for $i = 1, n$;

(2) on each subinterval $[R_i, R_{i+1}]$ for $i = 1, n-1$, p is a quintic in R;

(3) p has a continuous third derivative on the whole interval, $[R_1, R_n]$.

(4) The fourth and fifth derivatives of p, although continuous on any subinterval, (R_i, R_{i+1}), are, in general, discontinuous at R_i for $i = 2, n-2$. However, at all points, R_i for $i = 1, n-1$, a right-fourth and right-fifth derivative are well defined.

The above data are not sufficient,[4] we need values for f_1'' and f_n''. In the case of the Bevatron measurements these were provided by the cubic spline fits for $f(R)$, $R \leqslant R_1$, and $R \geqslant R_n$.[1] We set $p_1'' = f_1''$ and $p_n'' = f_n''$, then the properties (1), (2), and (3) lead to a tridiagonal linear system with diagonal dominance in the unknowns, p_i'', $i = 2, n-1$. After solution of this system we have for each R_i for $i = 1, n-1$,

$$p_i, p_i', p_i'', p_{i+1}, p_{i+1}', \text{ and } p_{i+1}'',$$

and these determine the quintic segment on the subinterval $[R_i, R_{i+1}]$. From this quintic segment we can, at any R_i for $i = 1, n-1$, compute the values of the third, right-fourth and right-fifth derivative. Then for any $R \in (R_1, R_n)$ we have $R \in [R_i, R_{i+1}]$ for some i. The values of p, p' and p'' at R can be readily obtained from the Taylor expansions (rightward) from R_i, such expansions being truncated after the term involving the fifth derivative.

4. COMPUTER CODES

Computer codes QUNMOD and QUYNSP have been written in Fortran 66 for the CDC 6600 which, respectively, smooth and fit the measured $f(R_i)$, $f'(R_i)$. A modification of the Bevatron tracking code, BEVORB, has been written which interpolates on the quintic fit. Since, for much of the Bevatron field, higher order data are not available, second order approximation is still retained in computing field components off the median plane.

5. CONCLUSION

The process described above has enabled us to make use of additional field information (gradients) where it is available. Smoothing of data is often subjective (possibly soundly based on *a priori* knowledge of function behaviour), but at least our method is definitive and consistent with our objective of quintic spline fitting. The effort to relieve even slight 'inconsistencies' in field and gradient measurements seems meritorious. Comparisons of tracking results obtained by the previous[1] method and this modification are still to be completed. On the negative side, there is always the question as to whether the quality and quantity of data available justify the employment of this more elegant method. Perhaps the only sound evaluation must be based on comparison of computed tracking results with observed results in the accelerator.

The decision to use linear programming techniques rather than least square minimization was based on the facility with which bounds could be placed on the f_i and f_i'. With bounds the least square formulation would be non-linear.

REFERENCES

1. J. Young, Differential Analysis of Magnetic Field Measurements with Applications, *Proceedings of the 1969 Particle Accelerator Conference* (F. T. Howard, Ed.), IEEE Trans. Nucl. Sci., **16**, No. 3, 762 (1969).
2. J. Young, Smoothing Data with Tolerances by Use of Linear Programming, Lawrence Radiation Laboratory Report, UCRL-19403, November 1969.
3. G. B. Dantzig, The Rand Corporation, Rand Report P-89, Santa Monica, California, July 1956.
4. J. Young, Function and Derivative Fitting by Modified Quintic Spline, Lawrence Radiation Laboratory Report, UCRL-19404, November 1969.

Received 8 April 1970